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Published in:
Indagationes mathematicae-New series

DOI:
10.1016/j.indag.2019.03.001

The Nullcone of the Lie Algebra of G_2

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January 25, 2019

Abstract

This paper investigates the nilpotent conjugacy classes of the Lie algebra of the simple algebraic group of type G_2. These classes are determined by first finding the stratification, and then finding the classes within the strata. Except for characteristic 3, the classes coincide with the strata. In characteristic 3, one stratum splits into two orbits. If the characteristic differs from 2 and 3, the classes are determined by the singularities of the nilpotent variety. In characteristic 3, the matter is undecided yet. In characteristic 2, different classes have the same singularities.

Key words: simple group; G_2; nilpotent; singularity; stratification.

1 Introduction

Let G be a reductive group with Lie algebra $\mathfrak{g}$. Can the nilpotent conjugacy classes in $\mathfrak{g}$ be characterized by the singularities of the nilpotent variety? In the paper [7], a positive answer to this question was given for the cases that G is $\mathbb{GL}(n)$, or $\mathbb{S}p(n)$ and $\mathrm{char}(K) \neq 2$. The primary aim of the present paper is to extend this result to the group G_2.

The first thing to do is to determine the nilpotent conjugacy classes in $\mathfrak{g}$. Traditionally, they are classified by means of the Theorem of Jacobson-Morozov. This leads, however, to unnatural assumptions on the characteristic of the field. Stuhler [24] was one of the first to determine the nilpotent conjugacy classes for G_2 in all characteristics. More recently, the book [13] determines these classes for all simple groups.

We propose a two-step approach for the determination of the nilpotent conjugacy classes for G_2. The first step is the determination of the stratification of the nullcone of $\mathfrak{g}$ in the sense of [9]. This can be done independently of the characteristic. In the second step, the orbits within the strata are determined. Here, the characteristics 2 and 3 need special attention. The stratification theory of [9] is presented here over a field of arbitrary characteristic, and is extended slightly for the sake of efficient computation.

There are two classical ways to construct G_2: either as the fixpoint set for the outer symmetry of the Dynkin diagram of D_4, or as the automorphism group of an octonion algebra. The group D_4 is most easily represented as $\mathbb{SO}(8)$. We can therefore combine these ways in an eight dimensional representation of G_2, except for a minor gap in characteristic 2.

In general, there are five nilpotent conjugacy classes (orbits): a regular class of dimension 12, a subregular class of dimension 10, the subsubregular class (dimension 8), the class of the long root vector (dimension 6), and the origin (dimension 0). In characteristic 3, however, the subsubregular class splits into two orbits of dimensions 8 and 6.

The nilpotent variety or nullcone is the zero set of the homogeneous invariant polynomials. For G_2, the ring of the invariant polynomials is generated by two homogeous elements, one of degree 2, and the other of degree 3 if $\mathrm{char}(K) = 2$, and of degree 6 otherwise.

In [7], the singularities in the nilpotent varieties of $\mathbb{GL}(n)$ and $\mathbb{S}p(n)$ are characterized by a numerical criterion ord*. This criterion is used here as well. It separates the orbits in the nilpotent variety of G_2 in characteristics $\neq 2, 3$. In characteristic 3, the singularities in the two subsubregular orbits seem to be different, but they are not separated by ord*. In characteristic 2, the nilpotent variety is smooth in the regular orbit and in the subregular orbit, while the other two nonzero orbits have singularities that are smoothly equivalent.

Overview

Section 2 deals with the general theory of the stratification of the nullcone, for a reductive group over an algebraically closed field of arbitrary characteristic. In Section 3 an eight dimensional representation of G_2 is constructed, more precisely of the split version of G_2 over an arbitrary field. Section 4 presents the stratification of the nullcone of the Lie algebra of G_2, followed by the determination of the orbit structure. In Section 5 the nilpotent variety is defined, and its singularities are related to the orbits.

2 The Stratification of the Nullcone

The stratification of the nullcone is based on classical ideas of Hilbert and Mumford [17], presented in Sections 2.1 and 2.2, and in particular on the optimality theory of Kempf [11], presented in the Sections 2.3 and 2.4.

The stratification theory of [9, 10] is presented in Section 2.5. This theory is extended here slightly to make it easier to determine the stratification in concrete cases. In Section 2.6 it is shown that the nullcone of the Lie algebra $\mathfrak{g}$ of the group G is the set of the nilpotent elements of $\mathfrak{g}$.

2.1 Concentration and the nullcone

Let K be an algebraically closed field of arbitrary characteristic. Let G be a linear algebraic group over K, cf. [1]. Let $X(G)$ denote the abelian group of the characters $\chi : G \to \mathbb{GL}(1)$, and let $Y(G)$ be the set of the homomorphisms $\lambda : \mathbb{GL}(1) \to G$. If $\chi \in X(G)$ and $\lambda \in Y(G)$ then $(\chi, \lambda) \in \mathbb{Z}$ is defined by $\chi(\lambda(t)) = t^{(\chi,\lambda)}$. If $\lambda \in Y(G)$ and $n \in \mathbb{Z}$ then $n\lambda \in Y(G)$ is defined by $(n\lambda)(t) = \lambda(t^n)$. If G is a torus, $X(G)$ and $Y(G)$ are free $\mathbb{Z}$-modules of finite rank and $(\, ,)$ defines a duality between them.

The set $M(G)$ is defined as the set of equivalence classes for the equivalence relation $\sim$ on $Y(G) \times \mathbb{N}_+$ where $(\mu, m) \sim (\nu, n)$ if and only if $n\mu = m\nu$. The elements of $M(G)$ are called *coweights*. If G is a torus, $M(G) = Y(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a vector space and $X(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is its dual.

Let V be a pointed affine G-variety, i.e., G acts on V and V has a G-invariant base point $*$. A point $v \in V$ is called *concentrated* if there is $\lambda \in Y(G)$ such that $\lim_{t\to 0} \lambda(t)(v) = *$. The assertion $\lim_{t\to 0} \lambda(t)v = *$ means that there is a morphism of algebraic varieties $f : \mathbb{A}^1 \to V$ with $f(0) = *$ and $f(t) = \lambda(t)v$ for $t \neq 0$. Mumford [17] defined $m(v, \lambda)$ to be the multiplicity of the fiber $f^{-1}(*)$. By convention $m(*, \lambda) = +\infty$. If $\lambda \in M(G)$ then $n\lambda \in Y(G)$ for some $n > 0$ and we can define $m(v, \lambda) = n^{-1}m(v, n\lambda)$. For rational r, the set $V(\lambda, r) = \{v \in V \mid r \leq m(v, \lambda)\}$ is a closed subset of V.

The *nullcone* $Nc(V)$ is defined as the set of concentrated points of V. In general, the nullcone need not be closed, see [10, 1.3].

2.2 The Hilbert-Mumford theory

Two central results in this area must be mentioned. The first one is Theorem A.1.0 of [17, p. 192]:

Theorem 1 *The algebraic group G is reductive (in the sense of [1]) if and only if, for every finitely generated K-algebra R on which G acts rationally by K-automorphisms, the ring of invariants R^G is finitely generated.*

The second result (given e.g. in [10, Section 1.2]) shows that concentration is closely related to invariant theory.

Theorem 2 *Assume that the algebraic group G is reductive. Let V be a pointed affine G-variety. For $v \in V$, the following three conditions are equivalent:*
(i) v is concentrated,
(ii) the point $*$ is in the closure of the orbit Gv,
(iii) $f(x) = f(*)$ for every G-invariant function f on V.

Let $A(V)$ be the K-algebra of the polynomial functions on V. The group G acts of $A(V)$ by K-automorphisms. If G is reductive, Theorem 1 implies that $A(V)^G$ is finitely generated, and Theorem 2 implies that the nullcone $Nc(V)$ is the zero-set of the ideal in $A(V)$ generated by the functions in $A(V)^G$ that vanish in the point $*$. In particular, the nullcone is closed.

2.3 Optimality

A *norm* q on $M(G)$ is a function $q : M(G) \to \mathbb{Q}$ such that

1. If $\lambda \in M(G)$ is nonzero then $q(\lambda) > 0$.
2. If $\lambda \in M(G)$ and $g \in G$ then $q(\mathrm{int}(g)\lambda) = q(\lambda)$.
3. If T is a subtorus of G the restriction of q to $M(T)$ is a quadratic form on the vector space $M(T)$, with an associated inner product such that $(\lambda, \lambda) = q(\lambda)$.

It is well known that a norm of $M(G)$ exists. More precisely, if T is a maximal torus of G and W is the Weyl group, every W-invariant norm on $M(T)$ has a unique extension to a norm of $M(G)$, and every norm on $M(G)$ restricts to a W-invariant norm on $M(T)$, see [8, 17].

From now, a norm q on $M(G)$ is fixed. If X is a subset of V, the number $q^*(X)$ is defined by

$$q^*(X) = \inf\{q(\lambda) \mid \lambda \in M(G) : X \subset V(\lambda, 1)\} \ .$$

The set X is said to be *concentrated* iff $q^*(X) < \infty$, i.e., if $X \subset V(\lambda, 1)$ for some λ. The *optimal class* $\Lambda(X)$ is defined by

$$\Lambda(X) = \{\lambda \in M(G) \mid X \subset V(\lambda, 1) \ \wedge \ q(\lambda) = q^*(X)\} \ .$$

For simplicity, we assume henceforward that V is a G-module pointed by 0. If T is a torus in G, let $V = \sum_\pi V_\pi$ be the corresponding weight space decomposition where π ranges over $X(T)$. For any subset R of $X(T)$, let $V[R] = \sum_{\pi \in R} V_\pi$. The *Newton polytope* $R(X, T)$ of X is defined as the smallest subset R of $X(T)$ with $X \subset V[R]$. For any $\lambda \in M(T)$, let

$$H(\lambda) = \{\pi \in X(T) \mid 1 \le (\pi, \lambda)\}\ ,$$
$$q^*(R) = \inf\{q(\lambda) \mid \lambda \in M(T) : R \subset H(\lambda)\}.$$

If R is a finite set with $q^*(R) < \infty$ then, by convexity, there is a unique coweight $\delta = \delta(R)$ with $q(\delta) = q^*(R)$ and $R \subset H(\delta)$ (the coweight δ is called the *minimal* coweight for R). We have $X \subset V(\lambda, 1)$ if and only if $R(X, T) \subset H(\lambda)$. Therefore $\lambda \in \Lambda(X) \cap M(T)$ implies $\lambda = \delta(R(X, T))$. This proves

Lemma 3 *Let X be a subset of V and let T be a torus in G. Then $\Lambda(X) \cap M(T)$ contains at most one element.*

Assume X is a concentrated set. Let T be a maximal torus of G. As all maximal tori of G are conjugate, it holds that

$$q^*(X) = \inf\{q^*(R(g^{-1}X, T)) \mid g \in G\}\ .$$

As all Newton polytopes for T are contained in the finite set $R(V, T)$, there exists $h \in G$ with

$$q^*(X) = q^*(R(h^{-1}X, T)) = q^*(R(X, \mathrm{int}(h)T))\ .$$

Putting $T_0 = \mathrm{int}(h)T$ and $\delta_0 = \delta(R(X, T_0))$, we have $\delta_0 \in \Lambda(X) \cap M(T_0)$. As the other implication is trivial, this proves

Lemma 4 *Let X be a subset of V. The set $\Lambda(X)$ is nonempty if and only if the set X is concentrated.*

2.4 Kempf's theorem

From now onward, the group G is assumed to be reductive and connected.

The interior action of G on itself, given by $\mathrm{int}(g)h = ghg^{-1}$ and pointed by $* = e$, is of particular importance. Because G is reductive, the corresponding subset $P(\lambda) = G(\lambda, 0)$ is a parabolic subgroup of G, e.g., by [17, p. 55]. If $\mu = \mathrm{int}(p)\lambda$ for some $p \in P(\lambda)$, then $V(\mu, r) = V(\lambda, r)$ for any pointed affine G-variety V; in particular $P(\mu) = P(\lambda)$. We therefore have an equivalence relation $\sim$ on $M(G)$ defined by $\lambda \sim \mu$ iff $\mu = \mathrm{int}(p)\lambda$ for some $p \in P(\lambda)$. The quotient set is called the *vectorial building* $\mathbf{Vb}(G) = (M(G)/{\sim})$. For a pointed affine G-variety V and $\Lambda \in \mathbf{Vb}(G)$, we can now define $V(\Lambda, r) = V(\lambda, r)$ for any $\lambda \in \Lambda$. In particular $P(\Lambda) = P(\lambda)$.

Lemma 5 *[11, 9]. Let Λ_1, $\Lambda_2 \in$ **Vb**(G) be such that $(\Lambda_1 \cup \Lambda_2) \cap M(T)$ contains at most one element for every torus T in G. Then $\Lambda_1 = \Lambda_2$.*

This result is used to prove Kempf's optimality theorem [11, 9]:

Theorem 6 *Let X be a concentrated set. Then $\Lambda(X)$ is a single equivalence class of $M(G)$, i.e. an element of* **Vb**(G).

Proof. If $\lambda \sim \mu$ then $V(\lambda, 1) = V(\mu, 1)$ and $q(\lambda) = q(\mu)$. Therefore $\Lambda(X)$ is a union of equivalence classes. Lemma 3 and Lemma 5 together imply that $\Lambda(X)$ is contained in one equivalence class. Lemma 4 says that $\Lambda(X)$ is nonempty. □

In view of the above, for concentrated set X, the *saturation* $S(X)$ and the *Kempf group* $P(X)$ of X are defined by $S(X) = V(\Lambda(X), 1)$ and $P(X) = P(\Lambda(X))$. The set $S(X)$ is a concentrated closed subset of V that contains X. The group $P(X)$ is a parabolic subgroup of G, and it is the stabilizer of $S(X)$, i.e. $P(X) = \{g \in G \mid gS(X) \subset S(X)\}$, cf. [9].

The functions q^*, Λ, S, P implicitly depend on the group G. If useful, an index G is used to make the dependence explicit.

For a not-necessarily reductive subgroup H of G, the set $M(H)$ is a subset of $M(G)$, and the restriction of q to $M(H)$ is a norm on $M(H)$. A subgroup H is called *optimal* for X iff $q^*_H(X) = q^*_G(X)$, in which case $\Lambda_H(X) = M(H) \cap \Lambda_G(X)$.

2.5 The stratification

For the stratification and the subsequent orbit classification we are interested mainly in the case that X is a singleton set $\{v\}$, in which case we write $q^*(v)$, $\Lambda(v)$, $P(v)$, $S(v)$ instead of $q^*(\{v\})$, $\Lambda(\{v\})$, etc.

The stratification of the nullcone is defined in [9] by means of the equivalence relations $\sim$ and $\approx$ on $Nc(V)$ given by

$$x \approx y \iff \Lambda(x) = \Lambda(y) \ ,$$
$$x \sim y \iff \Lambda(gx) = \Lambda(y) \text{ for some } g \in G.$$

An equivalence class $[v] = \{x \mid x \approx v\}$ is called a *blade*. An equivalence class $G[v] = \{x \mid x \sim v\}$ is called a *stratum*. In [9, 4.2], the following result is proved.

Proposition 7 *Let $v \in Nc(V)$.*
(a) $[v] = \{x \in S(v) \mid q^*(x) = q^*(v)\}$. It is open in $S(v)$ and $S(v)$ is its closure.
(b) $GS(v)$ is an irreducible closed subset V, contained in $Nc(V)$.
(c) $G[v] = \{x \in GS(v) \mid q^*(x) = q^*(v)\}$. It is open and dense in $GS(v)$.

A coweight λ is called *optimal* for V iff the set $b(V,\lambda) = \{x \in V \mid \lambda \in \Lambda(x)\}$ is nonempty, or equivalently iff $b(V,\lambda)$ is a blade.

In the next results, the theory of [9] is extended slightly to make it easier to determine the stratification. Fix a maximal torus T and a Borel group B of G with $T \subset B$. Let B_u be the maximal unipotent subgroup of B.

Proposition 8 *Let $\lambda \in M(T)$ and $v \in Nc(V)$. It holds that $\lambda \in \Lambda_G(v)$ if and only if $v \in V(\lambda, 1)$ and $q(\lambda) \le q(\mu)$ for every optimal coweight μ and every $g \in B_u$ with $gv \in V(\mu, 1)$.*

Proof. Assume $\lambda \in \Lambda_G(v)$. Then $v \in V(\lambda,1)$ by definition. Moreover $q(\lambda) = q^*(v) = q^*(gv) \le q(\mu)$ whenever $gv \in V(\mu,1)$ (and $g \in B_u$ and μ optimal).

Conversely, assume that $v \in V(\lambda,1)$ and $\lambda \notin \Lambda_G(v)$. Then $q^*(v) < q(\lambda)$. By [8, 5.4(b)], the parabolic group B is optimal for v. Therefore, B has an optimal coweight $\mu' \in M(B) \cap \Lambda(v)$ with $q(\mu') = q^*(v) < q(\lambda)$. Moreover B has a maximal torus T' with $\mu' \in M(T')$. As all maximal tori of B are conjugate under the unipotent group B_u, there is $g \in B_u$ with $\mathrm{int}(g)T' = T$. Then $\mu = \mathrm{int}(g)\mu' \in M(T) \cap \Lambda(gv)$ has $q(\mu) < q(\lambda)$ and $gv \in V(\mu,1)$. □

Although they are not equivalent, this proposition plays here the same role as the Kirwan-Ness criterion [12, 18] in e.g. [20] and [4].

Coming back to the concepts and notations of Section 2.3, a coweight $\lambda \in M(T)$ is called *preoptimal for V* iff $\lambda = \delta(H(\lambda) \cap R(V,T))$.

Lemma 9 *(a) Let $\lambda \in M(T)$ be optimal for V. Then it is a preoptimal for V.*
(b) In $M(T)$, the number of preoptimal coweights for V is finite.

Proof. (a) There is $v \in Nc(V)$ with $\lambda \in \Lambda(v)$. Then $\lambda = \delta(R(v,T))$. The set $R(v,T)$ is a subset of $H(\lambda) \cap R(V,T)$, and that the latter set has the same minimal coweight as $R(v,T)$. Therefore λ is a preoptimal.

(b) The set $R(V,T)$ is finite and has therefore only finitely many subsets. □

Recall that a coweight λ is called *dominant* iff $B \subset P(\lambda)$. A coweight λ is called *critical* iff it is both optimal and dominant. It is called a *candidate* iff it is both preoptimal and dominant.

It is easy to see that every blade U is a concentrated set and satisfies $\Lambda(U) = \Lambda(v)$ for all $v \in U$. A blade U is called *dominant* iff $B \subset P(\Lambda(U))$, or equivalently iff there is a critical coweight $\lambda \in \Lambda(U)$.

Lemma 10 *(a) If X is a stratum of* Nc(V) *there is a unique dominant blade U with $X = G \cdot U$.*
(b) If U is a dominant blade of Nc(V) *there is a unique critical coweight $\lambda \in M(T)$ with $U = b(V, \lambda)$.*
(c) Conversely, if λ is a critical coweight, then $b(V, \lambda)$ is a dominant blade and is open and dense in $V(\lambda, 1)$, the set $G \cdot b(V, \lambda)$ is a stratum and is open and dense in the closed set $G \cdot V(\lambda, 1)$.
(d) The strata of Nc(V) *are in bijective correspondence with the dominant blades, and with the critical coweights.*
(e) Nc(V) *has finitely many strata.*

Proof. (a) One can choose $v \in X$ with $B \subset P(v)$. The blade $[v]$ is dominant and satsfies $X = G \cdot [v]$. In order to prove uniqueness, assume $X = G \cdot U$ for some other dominant blade U. Then there is $g \in G$ with $gv \in U$. Put $P = P(v)$. Then $\mathrm{int}(g)P = P(gv)$. As both $[v]$ and $[gv]$ are dominant blades, B is a subset of both parabolic groups P and $\mathrm{int}(g)P$. Therefore, $\mathrm{int}(g)P = P$ by [1, 11.17]. It follows that $g \in P$ by [1, 11.16]. This proves that $\Lambda(gv) = \Lambda(v)$ and, hence, $[gv] = [v]$.

The parts (b), (c), (d) can be left to the reader.

(e) The number of critical coweights is finite because of Lemma 9. □

In view of the above, the determination of the strata of V begins with the determination of the candidate coweights. It is often convenient instead of the candidates to determine the candidate weight sets: a set of weights $R \subset X(T)$ is called a *candidate weight set* iff $\lambda = \delta(R)$ is dominant and satisfies $R = H(\lambda) \cap R(V, T)$ (so that λ is a candidate). A candidate weight set R is called *critical* iff $\delta(R)$ is critical.

Stratification is a step towards orbit classification because of

Lemma 11 *Let $v \in Nc(V)$. Then $Gv \cap [v] = P(v)v$.*

Proof. First let $w \in Gv \cap [v]$. Then there is $g \in G$ with $w = gv$. It follows that $g[v] = [w] = [v]$. Therefore $g \in P(v)$ and $w \in P(v)v$. The converse inclusion is trivial. □

In some cases, the stratum is a single orbit:

Lemma 12 *Let V be a G-module. Let $v \in Nc(V)$ have* $\dim(S(v)) = 1$*. Then $[v]$ is the $P(v)$-orbit of v, and $G[v]$ is the G-orbit of v.*

Proof. Choose $\lambda \in \Lambda(v)$. Then $\lambda(t)v \in S(v)$ for all t. It fills $S(v)$ because $S(v)$ has dimension 1. □

2.6 The nullcone and nilpotency

We now specialize to the case that V is the Lie algebra $\mathfrak{g}$ of G, which is a G-module for the adjoint action of G. It is well-known that the nullcone $Nc(\mathfrak{g})$ is the set *Nilp*(G) of the nilpotent elements of $\mathfrak{g}$, which is irreducible [21, (5.4)]. For lack of a suitable reference that applies to arbitrary characteristic, we provide the main arguments.

Lemma 13 *Let B be a Borel subgroup of G, with Lie algebra $\mathfrak{b}$. Let B_u be the maximal unipotent subgroup of B, with Lie algebra $\mathfrak{b}_u$. Then $Nilp(G) = Nc(\mathfrak{g}) = Ad(G)\mathfrak{b}_u$.*

Proof. The set *Nilp*(G) is a subset of $\mathrm{Ad}(G)\mathfrak{b}_u$ because, by [1, 14.25], every nilpotent element of $\mathfrak{g}$ is conjugate to a nilpotent element of $\mathfrak{b}$, i.e., to an element of $\mathfrak{b}_u$. The set $\mathrm{Ad}(G)\mathfrak{b}_u$ is contained in $Nc(\mathfrak{g})$ because $\mathfrak{b}_u$ is a concentrated set. The set $Nc(\mathfrak{g})$ is a subset of *Nilp*(G) because every concentrated element has all eigenvalues zero and is therefore nilpotent because of Cayley-Hamilton. □

3 The Construction of G_2

There are two classical ways to construct the simple group G_2 and its Lie algebra. One is as the fixed points of the outer automorphisms of the group D_4 induced by the symmetry of its Dynkin diagram. The other is as the automorphism group of an octonion algebra. Both constructions involve several choices. In order to enforce compatibility of the choices we begin with the approach via D_4 represented as $\mathbb{SO}(8)$; subsequently an octonion algebra is constructed in the associated eight dimensional vector space.

In this section, the field K can be arbitrary, it need not be algebraically closed. The quadratic form, however, is supposed to be split. For simplicity we do not use Clifford algebras and spinors, as in [23, Chapter 3]. Therefore, the argument has a small gap in characteristic 2.

3.1 The orthogonal group in eight dimensions

Let V be an eight dimensional vector space with a basis $b_0, \ldots, b_7$, over an arbitrary field K. Let the *norm* $N : V \to K$ be the quadratic form given by $N(\sum_i \xi_i b_i) = \xi_0\xi_7 + \xi_1\xi_6 + \xi_2\xi_5 + \xi_3\xi_4$. The associated bilinear form is given by $\langle x, y\rangle = N(x + y) - N(x) - N(y)$. Note that

$$(0) \qquad \langle b_i, b_j\rangle \neq 0 \quad \Longleftrightarrow \quad i + j = 7 \ .$$

The special orthogonal group $\mathbb{SO}(V)$ is the group of the linear transformations of V that preserve the norm N and have determinant 1. The Lie algebra $\mathfrak{so}(V)$ consists of the matrices Y that satisfy $\langle Yx, x\rangle = 0$ for all $x \in V$. With respect to the basis $b_0, \ldots, b_7$, the elements of $\mathfrak{so}(V)$ have matrices of the form

$$Y = \begin{pmatrix} c_{12} & c_1 & -c_4 & c_9 & -c_8 & -c_{10} & -c_{11} & 0 \\ -c_{26} & c_{13} & c_0 & c_5 & -c_6 & -c_7 & 0 & c_{11} \\ c_{23} & -c_{27} & c_{14} & c_2 & -c_3 & 0 & c_7 & c_{10} \\ -c_{18} & -c_{22} & -c_{25} & c_{15} & 0 & c_3 & c_6 & c_8 \\ c_{19} & c_{21} & c_{24} & 0 & -c_{15} & -c_2 & -c_5 & -c_9 \\ c_{17} & c_{20} & 0 & -c_{24} & c_{25} & -c_{14} & -c_0 & c_4 \\ c_{16} & 0 & -c_{20} & -c_{21} & c_{22} & c_{27} & -c_{13} & -c_1 \\ 0 & -c_{16} & -c_{17} & -c_{19} & c_{18} & -c_{23} & c_{26} & -c_{12} \end{pmatrix}$$

Here the indices and the signs are chosen carefully, in a not very natural way.

Let the matrices $Y_0, \ldots, Y_{27}$ be given by $Y = \sum_{i=0}^{27} c_i Y_i$. The indices in the matrix are chosen in such a way that $Y_0, \ldots, Y_{11}$ are upper triangular matrices, that $Y_{12}, \ldots, Y_{15}$ are diagonal matrices and $Y_{16}, \ldots, Y_{27}$ are lower triangular. The Lie products $H_i = [Y_{27-i}, Y_i]$ are diagonal matrices with $[H_i, Y_i] = 2Y_i$ if $0 \le i < 12$ or $16 \le i < 28$.

The group $\mathbb{SO}(V)$ has an *adjoint* action on its Lie algebra $\mathfrak{so}(V)$ given by $\mathrm{Ad}(g)(Y) = gYg^{-1}$ for $g \in \mathbb{SO}(V)$ and $Y \in \mathfrak{so}(V)$. Let T be the maximal torus of the group $\mathbb{SO}(V)$ that consists of the invertible diagonal matrices

$$t = \mathrm{diag}(t_0, t_1, t_2, t_3, t_3^{-1}, t_2^{-1}, t_1^{-1}, t_0^{-1}) \ .$$

Let $X(T)$ be the character group of T, written additively. The characters λ_i for $0 \le i < 4$ are given by $\lambda_i(t) = t_i$, and the characters α_i for $0 \le i < 12$ by $\alpha_i(t)Y_i = \mathrm{Ad}(t)Y_i$. Then $\alpha_0, \ldots, \alpha_{11}$ are the positive roots with respect to the Borel group of the upper triangular matrices in $\mathbb{SO}(V)$. Among these, the simple roots are $\alpha_0 = \lambda_1 - \lambda_2$, $\alpha_1 = \lambda_0 - \lambda_1$, $\alpha_2 = \lambda_2 - \lambda_3$, $\alpha_3 = \lambda_2 + \lambda_3$.

Let θ be the involutive linear transformation of V that interchanges the basis vectors b_i and b_{7-i} for $0 \le i < 4$. It is clear that $\theta \in \mathbb{SO}(V)$. Its adjoint action

Figure 1: The Dynkin diagrams D_4 and G_2

$\mathrm{Ad}(\theta)$ is the automorphism of $\mathfrak{so}(V)$ that interchanges Y_i and Y_{27-i} for $0 \le i < 12$ and multiplies the matrices $Y_{12}, \ldots, Y_{15}$ with -1.

Recall that, for a semisimple algebraic group G with Lie algebra $\mathfrak{g}$, with maximal torus T, and root system R, if $\mathrm{char}(K) = 0$, a *Chevalley system* [3, Chap. 8, §2] is a family $(X_\alpha)_{\alpha \in R}$ of vectors in $\mathfrak{g}$ such that $\mathrm{Ad}(t)X_\alpha = \alpha(t)X$ for all $t \in T$, that the elements $H_\alpha = [X_{-\alpha}, X_\alpha]$ satisfy $[H_\alpha, X_\alpha] = 2X_\alpha$, and that $\mathfrak{g}$ has an automorphism that interchanges X_α and $X_{-\alpha}$. If $\mathrm{char}(K) = 0$, the basis of $\mathfrak{o}(V)$ constructed above therefore induces a Chevalley system given by $X_{\alpha_i} = Y_i$ for $0 \le i < 12$ and $16 \le i < 28$.

Figure 1 shows the Dynkin diagram D_4 of $\mathbb{SO}(V)$. It consists of a central node 0 with three neighbours 1, 2, 3. The symmetry of the diagram allows for the permutations of the three neighbours. The fixed positive roots are α_0, $\alpha_{10} = \sum_{i=0}^{3} \alpha_i$, and $\alpha_{11} = \alpha_0 + \alpha_{10}$. The triples $(\alpha_1, \alpha_4, \alpha_7)$, $(\alpha_2, \alpha_5, \alpha_8)$, $(\alpha_3, \alpha_6, \alpha_9)$ are permuted as the nodes 1, 2, 3 of the Dynkin diagram.

This symmetry is extended to the root vectors Y_i. The minus signs in the above matrix of Y are chosen such that, if $0 \le i < j < k < 12$ and $\alpha_i + \alpha_j = \alpha_k$, then $[Y_i, Y_j] = Y_k$. Let $r \in \mathbb{O}(V)$ be the reflexion with $rb_3 = -b_4$, $rb_4 = -b_3$, and $rb_i = b_i$ for $i \ne 3$, 4. The adjoint action of r on $\mathfrak{so}(V)$ interchanges the pairs (Y_2, Y_3), (Y_5, Y_6), (Y_8, Y_9), (Y_{18}, Y_{19}), (Y_{21}, Y_{22}), (Y_{24}, Y_{25}), and $(Y_{15}, -Y_{15})$. It leaves the other basic matrices unchanged. In short, it interchanges the nodes 2 and 3 of the Dynkin diagram.

Now assume that $\mathrm{char}(K) \ne 2$. This assumption is needed to interchange the nodes 1 and 2 of the Dynkin diagram in the present representation of D_4. The root vectors (Y_1, Y_2), (Y_4, Y_5), (Y_7, Y_8), etc., are interchanged in the same way as in the previous case. To determine the transformation needed for the Lie algbra $\mathfrak{t}$ of the torus T, we now use the elements

$$\begin{aligned} H_0 &= [Y_{27}, Y_0] = Y_{13} - Y_{14} \ , \\ H_1 &= [Y_{26}, Y_1] = Y_{12} - Y_{13} \ , \\ H_2 &= [Y_{25}, Y_2] = Y_{14} - Y_{15} \ , \\ H_3 &= [Y_{24}, Y_3] = Y_{14} + Y_{15} \ . \end{aligned}$$

These vectors form a basis of $\mathfrak{t}$ because of $\mathrm{char}(K) \ne 2$. The interchange of the nodes 1 and 2 is completed by interchanging H_1 and H_2, and keeping H_0 and H_3 fixed.

We thus have constructed two automorphisms of the Lie algebra $\mathfrak{so}(V)$, which generate a group Γ isomorphic to the symmetric group of $\{1, 2, 3\}$. The fixed points of Γ in $\mathfrak{so}(V)$ of these automorphisms form a Lie algebra consisting of the matrices

$$X = \begin{pmatrix} a_{67} & a_0 & -a_2 & a_3 & -a_3 & -a_4 & -a_5 & 0 \\ -a_{13} & a_6 & a_1 & a_2 & -a_2 & -a_3 & 0 & a_5 \\ a_{11} & -a_{12} & a_7 & a_0 & -a_0 & 0 & a_3 & a_4 \\ -a_{10} & -a_{11} & -a_{13} & 0 & 0 & a_0 & a_2 & a_3 \\ a_{10} & a_{11} & a_{13} & 0 & 0 & -a_0 & -a_2 & -a_3 \\ a_9 & a_{10} & 0 & -a_{13} & a_{13} & -a_7 & -a_1 & a_2 \\ a_8 & 0 & -a_{10} & -a_{11} & a_{11} & a_{12} & -a_6 & -a_0 \\ 0 & -a_8 & -a_9 & -a_{10} & a_{10} & -a_{11} & a_{13} & -a_{67} \end{pmatrix}$$

where by convention $a_{67} = a_6 + a_7$. Compare [16, Section 7.2]. It is shown below that this is the Lie algebra of G_2, even in characteristic 2.

Remark. Another point of view is that the above approach is specialized to the case $K = \mathbb{Q}$. One then considers the basis of $\mathfrak{so}(V)$ consisting of the vectors H_0, H_1, H_2, H_3, and Y_i with $0 \le i < 12$ or $16 \le i < 28$, and the free $\mathbb{Z}$-module L generated by this basis. It turns out that L is a Lie algebra over $\mathbb{Z}$ with $\mathfrak{so}(V) = L \otimes_{\mathbb{Z}} \mathbb{Q}$. The above group Γ acts on L and the fixed points L^Γ form the Lie algebra of G_2 over $\mathbb{Z}$. Its elements are the matrices X with integer coefficients. □

3.2 A split octonion algebra

We turn to the second classical construction of G_2. Recall that a *composition algebra* C over a field K is a not necessarily associative algebra over K with unit element e, such that there is a quadratic function $N : C \to K$ that satisfies $N(xy) = N(x)N(y)$, and for which the associated bilinear form $\langle x, y\rangle = N(x+y) - N(x) - N(y)$ is nondegenerate.

A composition algebra C is called *split* if it has an *isotropic* vector, i.e., a nonzero vector $x \in C$ with $N(x) = 0$. All split composition algebras over K of the same dimension are isomorphic, cf. [23, Thm. 1.8.1]. If the base field K is algebraically closed, every composition algebra over it is split. The possible dimensions of composition algebras are 1, 2, 4, and 8. A composition algebra of dimension 8 is called an *octonion* algebra.

If G is the automorphism group of an octonion algebra C over K, it is a simple group of type G_2, see [23, Section 2.3]. Its Lie algebra consists of the derivations of the algebra, i.e., the linear transformations $D : C \to C$ with $D(xy) = (Dx)y + x(Dy)$.

Conversely, if one wants to define a multiplication on the vector space V of Section 3.1 in such a way that the matrices X of that section are derivations, one gets a system of homogeneous linear equations in 8^3 unknowns. This system has a five-dimensional solution space. Adding the requirement of a unit element $e = \sum_{i=0}^{7} e_i b_i$ with $ex = x$ and $xe = x$, one gets 8 more unknows and 128 additional equations. The final requirement $N(xy) = N(x)N(y)$ leads to four solutions.

One solution is chosen arbitrarily (we come back to the choice in Lemma 14 below). The solution can be described by so-called *vector matrices*, see [23, p. 20]. The vector $x = \sum_{i=0}^{7} \xi_i b_i$ in V is represented by the matrix

$$x = \begin{pmatrix} \xi_3 & (\xi_7, \xi_1, \xi_2) \\ (\xi_0, \xi_6, \xi_5) & \xi_4 \end{pmatrix}$$

The multiplication of such matrices is defined here by

$$\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix} \begin{pmatrix} \xi' & x' \\ y' & \eta' \end{pmatrix} = \begin{pmatrix} \xi\xi' - \langle x, y'\rangle & \xi x' + \eta' x - y \times y' \\ \eta y' + \xi' y - x \times x' & \eta\eta' - \langle y, x'\rangle \end{pmatrix}$$

where the three dimensional space K^3 has the inner product $\langle x, y\rangle$ given by

$$\langle(\xi_0, \xi_1, \xi_2), (\eta_0, \eta_1, \eta_2)\rangle = \textstyle\sum_{i=0}^{2} \xi_i \eta_i$$

and the outer product $x \times y$ given by

$$\langle x \times y, z \rangle = \det(x, y, z) \quad \text{for all } x,\ y,\ z \in K^3.$$

The full table of the multiplication in V is

	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7
b_0	0	0	0	b_0	0	b_1	$-b_2$	$-b_4$
b_1	0	0	$-b_0$	0	b_1	0	$-b_3$	b_5
b_2	0	b_0	0	0	b_2	$-b_3$	0	$-b_6$
b_3	0	b_1	b_2	b_3	0	0	0	b_7
b_4	b_0	0	0	0	b_4	b_5	b_6	0
b_5	$-b_1$	0	$-b_4$	b_5	0	0	b_7	0
b_6	b_2	$-b_4$	0	b_6	0	$-b_7$	0	0
b_7	$-b_3$	$-b_5$	b_6	0	b_7	0	0	0

By computer algebra one can verify that the vector $e = b_3 + b_4$ is the unit element of the algebra and that the norm is multiplicative, i.e. satisfies $N(xy) = N(x)N(y)$. Therefore the multiplication makes V an octonion algebra. As it is split and all split octonion algebras are isomorphic, it is the split octonion algebra. The multiplication is not commutative and not associative.

Let G_2 be the group of automorphisms of the octonion algebra V thus constructed, and let $\mathfrak{g}_2$ be its Lie algebra. It is known that $\dim \mathfrak{g}_2 = 14$. By computer algebra one verifies that the matrices X of the Section 3.1 are derivations of the octonion algebra V, and therefore elements of $\mathfrak{g}_2$. As the dimensions are equal, it follows that $\mathfrak{g}_2$ is the space of the matrices X. This argument applies even if the characteristic of the field K is 2.

The freedom of choosing a multiplication is explained by the following result.

Lemma 14 *Let $(V, \cdot, e, N)$ be a split octonion algebra and let $\mathfrak{g}_2$ be the Lie algebra of its derivations. Let Op be the set of the bilinear operators $\odot : V \times V \to V$ such that $(V, \odot, e', N)$ is an octonion algebra for some $e' \in V$, and that $Der(V, \odot) = \mathfrak{g}_2$. Then Op is the set of the operators $\odot_g$ given by $x \odot_g y = g^{-1}(gx \cdot gy)$ where g ranges over the centralizer C of $\mathfrak{g}_2$ in $\mathbb{O}(V)$, i.e., the group of the elements $g \in \mathbb{O}(V)$ with $Ad(g)X = X$ for all $X \in \mathfrak{g}_2$. If $\odot_g = \odot_h$ for $g,\ h \in C$, then $g = h$.*

Proof. Let $\odot \in Op$. Then $(V, \odot, e', N)$ is an octonion algebra for some unit $e' \in V$. This algebra has the same norm N as the first algebra and is therefore split. As all split octonion algebras are isomorphic, there is an isomorphism g from $(V, \odot, e', N)$ to $(V, \cdot, e, N)$. This means that $g(x \odot y) = gx \cdot gy$ for all $x,\ y \in V$, that $ge' = e$, and that $N(gx) = N(x)$ for all $x \in V$. This implies that $g \in \mathbb{O}(V)$, and that $x \odot y = g^{-1}(gx \cdot gy)$ for all $x,\ y \in V$.

Let $X \in \mathfrak{g}_2$. Then X is a derivation of $(V, \odot)$. So, for all $u,\ v \in V$, it holds that $X(u \odot v) = Xu \odot v + u \odot Xv$, or equivalently $Xg^{-1}(gu \cdot gv) = g^{-1}(gXu \cdot gv) + g^{-1}(gu \cdot gXv)$. Substituting $X' = gXg^{-1}$ and $u' = g^{-1}u$ and $v' = g^{-1}v$, we get that $X'(u' \cdot v') = Xu' \cdot v' + u' \cdot Xv'$. This holds for all u' and v', showing that $X' = Ad(g)X$ is a derivation of $(V, \cdot)$, i.e., $X \in \mathfrak{g}_2$. This holds for all $X \in \mathfrak{g}_2$. Therefore $\mathrm{Ad}(g)$ is an automorphism of $\mathfrak{g}_2$.

As every automorphism of the Lie algebra $\mathfrak{g}_2$ is an inner automorphism, it follows that the group G_2 has an element h with $\mathrm{Ad}(h)X = \mathrm{Ad}(g)X$ for all $X \in \mathfrak{g}_2$. Then $g_1 = h^{-1}g$ is an element of the centralizer and $x \odot y = g_1^{-1}(g_1 x \cdot g_1 y)$ for all x, $y \in V$.

Conversely, if g is in the centralizer, it is easy to see that $(V, \odot_g, g^{-1}e, N)$ is an octonion algebra with $\mathrm{Der}(V, \odot_g) = \mathfrak{g}_2$.

Finally, assume $\odot_g = \odot_h$ for $g,\ h \in C$. If we put $k = gh^{-1}$, it holds that $x \odot_k y = hg^{-1}(gh^{-1}x \cdot gh^{-1}y) = h(h^{-1}x \odot_g h^{-1}y) = h(h^{-1}x \odot_h h^{-1}y) = x \cdot y$. This

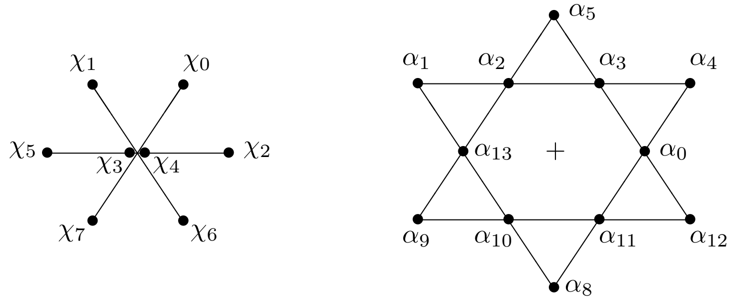

Figure 2: The weights of V and the root system of $\mathfrak{g}_2$.

implies that k preserves the multiplication and hence that $k \in G_2$. As $k \in C$, it acts trivial on the Lie algebra $\mathfrak{g}_2$. As the adjoint action of G_2 on its Lie algebra is known to be faithful, this implies $k = 1$ and, hence, $g = h$. □

The centralizer of $\mathfrak{g}_2$ in $\mathbb{O}(V)$ can be determined in the following way. One first determines the centralizer of $\mathfrak{g}_2$ in $\mathrm{End}(V)$. This consists of the matrices

$$g = \begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s & s-t & 0 & 0 & 0 \\ 0 & 0 & 0 & s-t & s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \end{pmatrix}$$

with $s, t \in K$. The centralizer in $\mathbb{O}(V)$ is obtained by intersecting with $\mathbb{O}(V)$. This boils down to the additional requirements $t^2 = 1$ and $s(s-t) = 0$. Therefore the centralizer of $\mathfrak{g}_2$ in $\mathbb{O}(V)$ is generated by -1 and the reflexion r mentioned in Section 3.1. It has four elements and is isomorphic to the multiplicative group $\{\pm 1\}^2$. By Lemma 14, it follows that there are four choices of an octonion algebra structure on V compatible with the chosen Lie algebra $\mathfrak{g}_2$.

3.3 The root system of G_2

Let T be the torus in G_2 of the diagonal matrices with respect to the basis $b_0, \ldots,$ b_7. Let the characters $\chi_i \in X(T)$ be given by $tb_i = \chi_i(t)b_i$. Formula (0) implies that $\chi_i(t)\chi_j(t) = 1$ when $i + j = 7$. Writing the character group $X(T)$ additively, it follows that $\chi_i + \chi_j = 0$ when $i + j = 7$. The identity $b_1b_2 = -b_0$ implies that $\chi_1 + \chi_2 = \chi_0$. The identities $b_3^2 = b_3$ and $b_4^2 = b_4$ imply that $\chi_3 = \chi_4 = 0$. Conversely, one can use the multiplication table to verify that the diagonal matrices $\mathrm{diag}(uv, u, v, 1, 1, v^{-1}, u^{-1}, u^{-1}v^{-1})$ with $u, v \neq 0$ are automorphisms of algebra V. This proves that T is a two-dimensional torus. In fact, it is a maximal torus in G_2, because any element $g \in G_2$ that commutes with T preserves the weight spaces, as well as the elements b_3 and b_4. The character group $X(T)$ is a free $\mathbb{Z}$-module with (e.g.) the basis χ_0, χ_1. The diagram of the weights $\chi_0, \ldots, \chi_7$ is drawn in the lefthand part of Figure 2. This diagram explains several zeroes in the multiplication table of V because, if $b_ib_j = \pm b_k$, the corresponding weights adds up: $\chi_i + \chi_j = \chi_k$.

The group G_2 preserves the unit element $e = b_3 + b_4$ and the norm N. It therefore also preserves the bilinear form of V.

As the Lie algebra $\mathfrak{g}_2$ consists of the matrices X of Section 3.1, it has the basis $X_0, \ldots X_{13}$, defined by the condition $X = \sum_i a_i X_i$. Then, e.g., it holds that $X_0 = Y_1 + Y_2 + Y_3$, $X_1 = Y_0$, etc. As before, the indices are chosen in such a way that the matrices $X_0, \ldots, X_5$ are upper triangular, that X_6 and X_7 are diagonal, and that $X_8, \ldots, X_{13}$ are lower triangular. Again, the Lie products $H_i = [X_{13-i}, X_i]$ satisfy $[H_i, X_i] = 2X_i$ if $0 \le i < 6$ or $8 \le i < 14$.

The elements $X_0, \ldots X_5$ are eigenvectors for the adjoint action of the torus T on the Lie algebra, with the respective weights $\alpha_0, \ldots, \alpha_5$ given by $\alpha_0 = \chi_2$, $\alpha_1 = \chi_1 - \chi_2$, and $\alpha_2 = \alpha_0 + \alpha_1$, $\alpha_3 = 2\alpha_0 + \alpha_1$, $\alpha_4 = 3\alpha_0 + \alpha_1$, $\alpha_5 = 3\alpha_0 + 2\alpha_1$. These weights form the set R_+. Similarly, the elements X_{13-i} $(0 \le i < 6)$ are eigenvectors for T with weights $-\alpha_i$ for all i with $0 \le i < 6$. Then $R = \{\pm\beta \mid \beta \in R_+\}$ is a root system of type G_2, with positive system R_+, and simple roots α_0, α_1. It is drawn as a six-pointed star in Figure 2.

The transformation θ used in Section 3.1 is an automorphism of the octonion algebra V and hence an element of the group G_2. The adjoint action $\mathrm{Ad}(\theta)$ of θ is the automorphism of $\mathfrak{g}_2$ that interchanges X_i and X_{13-i} for $0 \le i < 6$ and multiplies the diagonal matrices X_6, X_7 with -1. If $\mathrm{char}(K) = 0$, the basis therefore induces a Chevalley system of $\mathfrak{g}_2$, just as in Section 3.1.

Now that we have the weight space decomposition of the Lie algebra $\mathfrak{g}_2$, we can also form the corresponding one-dimensional subgroups U_β of the group G_2, cf. [1, Theorem (13.18)]. These are obtained by truncated exponential functions $g_i : K \to G_2$. For example, $g_1(u)$ is the transformation $1_C + uX_1$ of V, and a similar expression works for the other long roots. The function g_1 can be extended to a homomorphism $h_1 : \mathbb{SL}(2) \to G_2$ given by

$$h_1 \begin{pmatrix} x & z \\ y & t \end{pmatrix} = \mathrm{diag}\left(1, \begin{pmatrix} x & z \\ y & t \end{pmatrix}, 1, 1, \begin{pmatrix} x & -z \\ -y & t \end{pmatrix}, 1\right)$$

The short roots need the quadratic term of the exponential function. For example, $g_0(u)$ is the transformation $1_C + uX_0 + \frac{1}{2}u^2 X_0^2$. Strictly speaking, this expression requires $\mathrm{char}(K) \ne 2$, but by evaluating the matrix X_0^2 one gets a factor 2 which can formally cancel the factor $\frac{1}{2}$. Therefore, with some care, the expression turns out to work in characteristic 2 as well. Indeed the function can be extended to a homomorphism $h_0 : \mathbb{SL}(2) \to G_2$ given by

$$h_0 \begin{pmatrix} x & z \\ y & t \end{pmatrix} = \begin{pmatrix} x & z & 0 & 0 & 0 & 0 & 0 & 0 \\ y & t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x^2 & xz & -xz & z^2 & 0 & 0 \\ 0 & 0 & xy & xt & -yz & zt & 0 & 0 \\ 0 & 0 & -xy & -yz & xt & -zt & 0 & 0 \\ 0 & 0 & y^2 & yt & -yt & t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & -z \\ 0 & 0 & 0 & 0 & 0 & 0 & -y & t \end{pmatrix}$$

It follows that the function $f : K^6 \to G_2$ given by $f(u) = \prod_{i=0}^5 g_i(u_i)$ is an isomorphism of varieties between K^6 and the unipotent radical B_u of the upper-diagonal Borel group B of G_2, see [22, 10.1.1]. In other words, every element $b \in B_u$ is in a unique way a product $b = \prod_i g_i(u_i)$, where i ranges from 0 to 5 in some specified order. We use the clockwise order 1, 2, 5, 3, 4, 0 to fix the notation.

3.4 The nullcone of the octonions

As now the results of Section 2 are to be applied, assume that the field K is algebraically closed. The nullcone $Nc(V)$ of the octonion algebra V for the action of G_2 has a simple structure:

Theorem 15 *(a)* $Nc(V) = \{x \in V \mid \langle x, e\rangle = N(x) = 0\}$.
(b) $\dim(Nc(V)) = 6$.
(c) The nonzero elements of $Nc(V)$ *form a single* G_2*-orbit.*
(d) $Nc(V) = \{x \in V \mid x^2 = 0\}$.

Proof. It is easy to see that $Nc(V)$ is contained in the set $X = \{x \in V \mid \langle x, e\rangle = N(x) = 0\}$, and that $\dim(X) = 6$. It is clear that $b_0 \in Nc(V)$. Let x be an arbitrary nonzero element of $Nc(V)$. We have $\dim G_2 - \dim(P(x)) + \dim(S(x)) = \dim G_2[x] \leq 6$ by [9, 4.5(c)], and hence $\dim(S(x)) \leq \dim(P(x)) - 8$. As all proper parabolic subgroups of G_2 have dimension 8 or 9, this proves that $\dim(S(x)) = 1$. Moreover, in this case $\dim(G_2[x]) = 6$. The set X is irreducible, and therefore equals the closure of $G_2[x]$. This proves the parts (a) and (b). Part (c) follows from $\dim(S(x)) = 1$ and Lemma 12.

(d) By [23, Prop. 1.2.3], in any composition algebra, squaring satisfies

$$x^2 = \langle x, e\rangle x - N(x)e \ .$$

By (a), this formula implies that every element $x \in Nc(V)$ satisfies $x^2 = 0$. Conversely, if $x^2 = 0$, the formula implies that $x \in Nc(V)$ unless x is a multiple of e; the latter case is easily dealt with. □

It is possible to give a direct proof of part (c) using the results of Section 3.3.

Lemma 16 *The action of the unipotent radical* B_u *of the Borel group on* $Nc(V)$ *restricts to a simply transitive action on the intersection of* $Nc(V)$ *with* $b_7 + \sum_{i<7} Kb_i$.

Proof. Let $v = b_7 + \sum_{i<7} v_i b_i$ be an arbitrary element of the intersection. The claim is the unique existence of $g \in B_u$ with $gb_7 = v$. If one uses the parametrization of B_u given at the end of Section 3.3, the image $\prod_{i<6} g_i(u_i) b_7$ equals

$$\begin{aligned}&(u_0u_2u_3 + u_0u_5 - u_2u_4 + u_3^2, -u_0u_1u_3 - u_0u_2^2 + u_1u_4 + 2u_2u_3 + u_5,\\ &-u_0u_3 + u_4, -u_0u_2 + u_3, u_0u_2 - u_3, u_0u_1 + u_2, -u_0, 1) \ .\end{aligned}$$

This leads to unique values for $u_0, \ \ldots, \ u_5$. Finally, one uses the equalities of Theorem 15(a). □

Theorem 17 *The nonzero elements of* $Nc(V)$ *form a single orbit under the action of the group* G_2.

Proof. Let $v = \sum_{i<8} v_i b_i$ in $Nc(V)$ be nonzero The aim is to show that v is in the orbit of b_7. Because of Theorem 15(a), there is an index $i \in \{1, 2, 3, 5, 6, 7\}$ with $v_i \neq 0$. The elements

$$h_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad h_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

introduced in Section 3.3 can be used to interchange the coefficients v_0, v_1, v_2 and v_7, v_6, v_5. We may therefore assume that v_i is nonzero for some index $i \in \{0, 7\}$. The element $\theta \in G_2$ can be used to interchange v_0 and v_7. We may therefore assume that $v_7 \neq 0$. The torus can be used to normalize v_7. Therefore, we may assume that $v_7 = 1$, i.e., that $v \in b_7 + \sum_{i<7} Kb_i$. Finally, apply Lemma 16. □

4 The Nilpotent Conjugacy Classes of G_2

In this section, we determine the stratification and the orbit structure of the nullcone of the Lie algebra of G_2 over an algebraically closed field of arbitrary characteristic.

In characteristic 0, the orbits (conjugacy classes) of the nilpotent elements of the Lie algebra of G_2 are known for more than 60 years. This is briefly explained in Section 4.1. The stratification of the nullcone is determined in Section 4.2. It turns out that, in characteristic 0, the strata are the nilpotent orbits. In Section 4.3, differential methods are used to obtain concrete information on the relations between strata and orbits. Section 4.4 treats the two remaining orbits, and draws the conclusion.

4.1 The orbits in characteristic 0

Let G be a semisimple connected algebraic group over a field K of characteristic 0, with Lie algebra $\mathfrak{g}$. An $\mathfrak{sl}_2$-triplet in a Lie algebra $\mathfrak{g}$, is a triple (x, h, y) of elements of $\mathfrak{g}$ such that $[h, x] = 2x$, $[h, y] = -2y$, and $[y, x] = h$. The Theorem of Jacobson-Morozov [3, Chap. 8, §11] asserts that, if x is a nilpotent element of $\mathfrak{g}$, there exist elements h and y, such that (x, h, y) is an $\mathfrak{sl}_2$-triplet. Moreover, if h' and y' are such that (x, h', y') is another $\mathfrak{sl}_2$-triplet, there is an element $g \in G$ such that $\mathrm{Ad}(g)x = x$, $\mathrm{Ad}(g)h' = h$ and $\mathrm{Ad}(g)y' = y$. Strictly speaking, the book [3] disallows the $\mathfrak{sl}_2$-triplet $(0, 0, 0)$, but the extension to this case is trivial.

Following Dynkin [6], Springer and Steinberg [2, Part E, III, §4] apply $\mathfrak{sl}_2$-triplets to the classification of the nilpotent conjugacy classes in $\mathfrak{g}$. The result is as follows.

Assume that $\mathfrak{g}$ is split. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, let R be the corresponding root system, and let Δ be a basis of R. An $\mathfrak{sl}_2$-triplet (x, h, y) is called *normalized* iff $h \in \mathfrak{h}$ and that $\alpha(h) \in \{0, 1, 2\}$ for all $\alpha \in \Delta$. For every nilpotent conjugacy class $\mathcal{C}$, one can choose a normalized $\mathfrak{sl}_2$-triplet (x, h, y) with $x \in \mathcal{C}$. The Dynkin diagram $D(\mathcal{C})$ of $\mathcal{C}$ is defined as the Dynkin diagram of $\mathfrak{g}$ with numbers $\alpha(h)$ attached to the nodes $\alpha \in \Delta$. It is proved that $D(\mathcal{C})$ is uniquely determined by $\mathcal{C}$, and that classes $\mathcal{C}$ and $\mathcal{C}'$ are equal if and only if $D(\mathcal{C}) = \mathcal{D}(\mathcal{C}')$. In fact, Springer and Steinberg extend some of these results to some positive characteristics, but we do not persue this here.

Following Dynkin [6], we have the following table of normalized $\mathfrak{sl}_2$-triplets of the Lie algebra of G_2 over a field of characteristic 0, corresponding to the five nilpotent conjugacy classes.

α_0	α_1	rep	co	description	dim	ord*
2	2	$X_0 + X_1$	$10X_{12} + 6X_{13}$	regular	12	( )
0	2	$X_1 + X_4$	$2X_9 + 2X_{12}$	subregular	10	(1)
1	0	X_3	X_{10}	short root	8	(1, 1)
0	1	X_5	X_8	long root	6	(1, 1, 1)
0	0	0	0	zero	0	(2, 1, 1, 1, 1)

The first two columns give the Dynkin diagram, the numbers $\alpha(h)$. The column "rep" gives a representative x of the nilpotent class in terms of the basis of $\mathfrak{g}$ constructed in Section 3.3. The column "co" gives a Jacobson-Morozov companion y. The column "dim" gives the dimension of the class or its closure. The column "ord*" describes the singularity of *Nilp*(G) at an element of the class as determined below in Section 5.4.

4.2 Stratification in arbitrary characteristic

The theory of Section 2 is applied to the adjoint action of the group G_2 on its Lie algebra $\mathfrak{g}_2$. All norms on $M(G_2)$ are equivalent because G_2 is simple. We use the representation and the coordinates of Section 3.3. Let B be the Borel group of G_2 that corresponds to the positive system R_+, and let $T \subset B$ be the torus of the diagonal matrices. By inspection of the root system one easily obtains

Lemma 18 *The candidate weight sets of* $\mathfrak{g}_2$ *are:* $R_0 = \emptyset$, $R_1 = \{\alpha_5\}$, $R_2 = \{\alpha_3, \alpha_4, \alpha_5\}$, $R_3 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, $R_4 = R_+$, $R_5 = \{\alpha_4, \alpha_5\}$, *and* $R_6 = \{\alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

Lemma 19 *The sets* R_5 *and* R_6 *of Lemma 18 are noncritical.*

Proof. Let $v \in \mathfrak{g}_2$ with $\delta(R_5) \in \Lambda_T(v)$. Then $v = \xi X_4 + \eta X_5$ for some nonzero scalars ξ, η. Now the group element $g = g_1(-\eta/\xi)$, introduced at the end of Section 3.3, satisfies $\mathrm{Ad}(g)v = \xi X_4$. This implies $q^*(\mathrm{Ad}(g)v) < q^*(v)$, contradicting optimality of $\delta(R_5)$. This shows that R_5 is noncritical.

Let $v \in \mathfrak{g}_2$ with $\delta(R_6) \in \Lambda_T(v)$. Let U_2 be the span of X_3, X_4, X_5. Then $v \in \xi X_0 + \eta X_2 + U_2$ for some nonzero scalars ξ, η. Again the group element $g_1(-\eta/\xi)$ is used to eliminate X_2, giving a contradiction with optimality of R_6. This proves that R_6 is noncritical. □

Let the subspaces U_i $(0 \le i \le 4)$ be defined by $U_i = \mathfrak{g}_2[R_i]$. Putting $\delta_i = \delta(R_i)$, we have $U_i = \mathfrak{g}_2(\delta_i, 1)$. Note that the sets R_i with $0 \le i \le 4$ are ordered in such a way that $R_{i-1} \subset R_i$ and $q(\delta_{i-1}) < q(\delta_i)$ for $0 < i \le 4$. Put $U_i^o = b(\mathfrak{g}_2, \delta_i)$.

Theorem 20 *The dominant blades of* $Nc(\mathfrak{g}_2)$ *are the sets* U_i^o *for* $0 \le i \le 4$, *with the elements* $0 \in U_0^o$, $X_5 \in U_1^o$, $X_3 \in U_2^o$, $X_1 + X_4 \in U_3^o$, *and* $X_0 + X_1 \in U_4^o$. *For* $0 \le i \le 4$, *the set* U_i^o *is open and dense in* U_i.

Proof. As, by the Lemmas 18 and 19, all critical coweights are in the set $\{\delta_0, \delta_1, \delta_2, \delta_3, \delta_4\}$, it follows from Lemma 10 that the dominant blades of $Nc(\mathfrak{g})$ are the nonempty sets among U_i^o for $0 \le i \le 4$. It therefore remains to verify that the sets contain the elements claimed.

As $\delta_0 = 0$, it holds that $0 \in b(\mathfrak{g}_2, \delta_0) = U_0^o$. In the other four cases, Proposition 8 is applied. We have $X_5 \in \mathfrak{g}_2(\delta_1, 1)$. As $\mathrm{Ad}(b)X_5 = X_5$ for every $b \in B_u$, Proposition 8 gives $\delta_1 \in \Lambda_G(X_5)$. It follows that $X_5 \in U_1^o$.

It is clear that $X_3 \in U_2 = \mathfrak{g}_2(\delta_2, 1)$. For every $b \in B_u$, it holds that $\mathrm{Ad}(b)X_3 \in X_3 + KX_4 + KX_5$. This implies that $q_T^*(\mathrm{Ad}(b)X_3) = q(\delta_2)$. Proposition 8 gives $\delta_2 \in \Lambda_G(X_3)$. It follows that $X_3 \in U_2^o$.

Similarly, $X_1 + X_4 \in U_3 = \mathfrak{g}_2(\delta_3, 1)$. For every $b \in B_u$, it holds that

$$\mathrm{Ad}(b)(X_1 + X_4) = X_1 - \xi X_2 - \xi^2 X_3 + (1 - \xi^3) X_4 + \eta X_5$$

for some ξ, $\eta \in K$. This implies that $q_T^*(\mathrm{Ad}(b)(X_1 + X_4) = q(\delta_3)$ because the coefficient of X_1 and the coefficient of X_3 or X_4 is nonzero. Proposition 8 gives $\delta_3 \in \Lambda_G(X_1 + X_4)$. It follows that $X_1 + X_4 \in U_3^o$.

The proof of $X_0 + X_1 \in U_4^o$ is similar but simpler. □

The dimensions of the strata $G_2 \cdot U_i^o$ are determined with the formula $\dim(G_2 \cdot [v]) = \dim G_2 - \dim P(v) + \dim S(v)$. This gives the dimensions 0, 6, 8, 10, 12, respectively. The table of Section 4.1 thus extends nicely to arbitrary characteristic if one replaces conjugacy class by stratum, and ignores the numbers $\alpha(h)$ and the companions.

In our view the above determination of the stratification of $Nc(\mathfrak{g}_2)$ is simpler and more elementary than the methods of [4]. It shows that the stratification is independent of the characteristic of the field, confirming the results of [4]. In principle, our methods can be used for any simple group and pointed affine G-variety V but, in every case, the calculational bottleneck is the action of the Borel group of G on V.

In [14, 15], G. Lusztig proposed a definition of nilpotent pieces which leads to a stratification of the nullcone. According to [4, Remark 1 in Section 7.3], in the case of the classical groups, this stratification coincides with the stratification determined here. Our results may make it possible to see if the same idea works for the group G_2. This is a matter of future research.

4.3 Open orbits

Let the stabilizer P_i of U_i in G_2 have Lie algebra $\mathfrak{p}_i$. An element $v \in U_i$ has an open P_i-orbit in U_i if the tangent mapping $d\rho$ of the action $\rho : P_i \to U_i$ is surjective. This tangent mapping satisfies $d\rho(X) = \mathrm{ad}(X)(v) = -\mathrm{ad}(v)(X)$. Surjectivity of $d\rho$ is therefore equivalent to the condition that $\mathrm{ad}(v)$ has rank equal to $\dim(U_i)$. In each separate case the matrix of $\mathrm{ad}(v)$ is a submatrix of the 8 by 10 matrix of $\mathrm{ad}(v) : \mathfrak{b} + KX_{12} + KX_{13} \to \mathfrak{b}$, where $\mathfrak{b} = \sum_{i=0}^{7} KX_i$ is the Lie algebra of the Borel group. If $v = \sum_{i=0}^{5} v_i X_i$, the matrix is

$$\left(\begin{array}{cccccccc|cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -v_0 & v_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -v_1 & v_1 & 0 & -3v_2 \\
v_1 & -v_0 & 0 & 0 & 0 & 0 & -v_2 & 0 & 0 & 2v_3 \\
-2v_2 & 0 & 2v_0 & 0 & 0 & 0 & -v_3 & -v_3 & 0 & v_4 \\
-3v_3 & 0 & 0 & 3v_0 & 0 & 0 & -v_4 & -2v_4 & v_5 & 0 \\
0 & -v_4 & -3v_3 & 3v_2 & v_1 & 0 & -2v_5 & -v_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -v_1 & v_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_1 & -2v_0
\end{array}\right)$$

with respect to the basis $X_0, \ldots, X_7, X_{12}, X_{13}$.

In the case of U_4 the stabilizer P_4 is the Borel group with Lie algebra $\mathfrak{b}$. The matrix of $\mathrm{ad}(v) : \mathfrak{b} \to U_4$ is

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -v_0 \\
0 & 0 & 0 & 0 & 0 & 0 & -v_1 & v_1 \\
v_1 & -v_0 & 0 & 0 & 0 & 0 & -v_2 & 0 \\
-2v_2 & 0 & 2v_0 & 0 & 0 & 0 & -v_3 & -v_3 \\
-3v_3 & 0 & 0 & 3v_0 & 0 & 0 & -v_4 & -2v_4 \\
0 & -v_4 & -3v_3 & 3v_2 & v_1 & 0 & -2v_5 & -v_5
\end{pmatrix}$$

This matrix has rank 6 if and only if $6v_0v_1 \neq 0$.

In the case of U_3 the stabilizer P_3 has the Lie algebra $\mathfrak{p}_3 = \mathfrak{b} + KX_{13}$. The matrix of $\mathrm{ad}(v) : \mathfrak{p}_3 \to U_3$ for $v = \sum_{i=1}^{4} v_i X_i \in U_3$ with respect to the appropriate basis vectors X_i is

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -v_1 & v_1 & -3v_2 \\
v_1 & 0 & 0 & 0 & 0 & 0 & -v_2 & 0 & 2v_3 \\
-2v_2 & 0 & 0 & 0 & 0 & 0 & -v_3 & -v_3 & v_4 \\
-3v_3 & 0 & 0 & 0 & 0 & 0 & -v_4 & -2v_4 & 0 \\
0 & -v_4 & -3v_3 & 3v_2 & v_1 & 0 & -2v_5 & -v_5 & 0
\end{pmatrix}$$

This matrix has rank 5 if and only if $3(v_1^2v_4^2 + 6v_1v_2v_3v_4 - 4v_1v_3^3 + 4v_2^3v_4 - 3v_2^2v_3^2) \neq 0$.

In the case of U_2 the stabilizer P_2 has the Lie algebra $\mathfrak{p}_2 = \mathfrak{b} + KX_{12}$. The matrix of $\mathrm{ad}(v) : \mathfrak{p}_2 \to U_2$ for $v = \sum_{i=3}^{5} v_i X_i \in U_2$ with respect to the appropriate basis vectors X_i is

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -v_3 & -v_3 & 0 \\
-3v_3 & 0 & 0 & 0 & 0 & 0 & -v_4 & -2v_4 & v_5 \\
0 & -v_4 & -3v_3 & 0 & 0 & 0 & -2v_5 & -v_5 & 0
\end{pmatrix}$$

This matrix has rank 3 if and only if $v_3 \neq 0$, and $3 \neq 0$ or $v_4 \neq 0$ or $v_5 \neq 0$.

In view of these rank computations, we define on each of the spaces U_i for $0 \leq i \leq 4$ a polynomial f_i, viz. $f_0 = 1$, $f_1 = v_5$, $f_2 = v_3$, $f_4 = v_0v_1$, and

$$f_3 = v_1^2v_4^2 + 6v_1v_2v_3v_4 - 4v_1v_3^3 + 4v_2^3v_4 - 3v_2^2v_3^2 \; .$$

Here, $v_0, \ldots,$ are used as coordinates in the subspaces U_i. Let the zerosets be defined as $C_i = \{u \in U_i \mid f_i(u) = 0\}$.

Remark. The polynomial f_3 is the main invariant of the $\mathbb{SL}(2)$-module dual to the module of the cubic forms. If $\mathrm{char}(K) \neq 3$, the module of the cubic forms is self-dual so that f_3 is equivalent to the discriminant. This is not the case for $\mathrm{char}(K) = 3$. Anyhow, f_3 can be called the codiscriminant. $\square$

Lemma 21 *(a) If char$(K) \neq 2$, 3, then $U_4 \setminus C_4$ is a single orbit for P_4.*
(b) If char$(K) \neq 3$, then $U_3 \setminus C_3$ is a single orbit for P_3.
(c) If char$(K) \neq 3$, then $U_2 \setminus C_2$ is a single orbit for P_2.
(d) Assume that char$(K) = 3$. Then $U_2 \setminus C_2$ is the union of the P_2-orbits $U_{2a} = U_2 \setminus KX_3$ and $U_{2b} = \{tX_3 \mid t \neq 0\}$.
(e) $U_1 \setminus C_1$ is always a single P_1-orbit.

Proof. (a) As $\mathrm{char}(K) \neq 2$, 3, the tangent map at every element $v \in U_4 \setminus C_4$ is surjective, so that v is an interior point of its P_4-orbit in U_4. As U_4 is irreducible, it follows that all elements of $U_4 \setminus C_4$ are conjugate under P_4. At every point $v \in C_4$, the tangent map is not surjective. Therefore, these points are not conjugate to the points in $U_4 \setminus C_4$. The proofs for the cases (b) and (c) are similar.

(d) The argument used in the proofs of (a), (b), (c) also shows that U_{2a} is a single P_2-orbit. It easily follows that U_{2b} is a single P_2-orbit.

(e) Follows from Lemma 12. $\square$

Independent of the characteristic of the field we have

Lemma 22 *(a) $U_i^o = U_i \setminus C_i$ for $1 \leq i \leq 4$.*
(b) $U_0^o = U_0 = \{0\}$.

Proof. Let $i = 4$. Every element $v \in U_4 \setminus C_4$ satisfies $\delta(R(v,T)) = \delta_4$, and the set $U_4 \setminus C_4$ is invariant under conjugation by the group B_u. By Proposition 8 this implies that $U_4 \setminus C_4 \subset U_4^o$. On the other hand, the set C_4 is the union of the sets U_3 and $\mathfrak{g}[R_6]$. These two sets have optimal coweights smaller than δ_4. Therefore C_4 is disjoint with U_4^o. Together, this proves $U_4 \setminus C_4 = U_4^o$.

Let $i = 2$. Every element $v \in U_2 \setminus C_2$ satisfies $\delta(R(v,T)) = \delta_2$, and the set $U_2 \setminus C_2$ is invariant under conjugation by the group B_u. By Proposition 8 this implies that $U_2 \setminus C_2 \subset U_2^o$. On the other hand, the set C_2 equals $\mathfrak{g}[R_5]$ and this set has an optimal coweight smaller than δ_2. Therefore C_2 is disjoint with U_2^o. Together, this proves $U_2 \setminus C_2 = U_2^o$.

The treatment of the cases $i = 1$ and $i = 0$ is simpler and can be left to the reader.

It remains to treat $i = 3$. The function f_3 is invariant under the adjoint action of B_u. Therefore C_3 and its complement are invariant under B_u. If $v \in U_3 \setminus C_3$ then $a_1 \neq 0$ or $a_2 \neq 0$, and also $a_3 \neq 0$ or $a_4 \neq 0$. Therefore $R(v,T)$ contains α_1 or α_2, and also α_3 or α_4. It follows that $\delta(R(v,T)) = \delta_3$. As $U_3 \setminus C_3$ is invariant under B_u, Proposition 8 implies that $U_3 \setminus C_3 \subset U_3^o$.

It remains to prove that the set C_3 and U_3^o are disjoint. This is quite complicated. Let $v \in C_3$ be arbitrary. We have to show that $v \notin U_3^o$. We use the coordinates v_1, …, v_5 as above. If $v_1 = v_2 = 0$, then $v \in U_2$ and hence $v \notin U_3^0$. Therefore, assume $v_1 \neq 0$ or $v_2 \neq 0$. Let $U_2' \subset \mathfrak{g}_2$ be the span of the basis vectors X_1, X_2, X_5. The set U_2' is a conjugate of U_2, so that $q(\delta(U_2')) < q(\delta_3)$. By Proposition 8, it suffices to show that $\mathrm{Ad}(g)v \in U_2'$ for some $g \in B_u$. The one-dimensional subgroup g_0 is used for this purpose. The action of g_0 on U_3 is given by

$$g_0(\xi)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 - v_1\xi \\ v_3 + 2v_2\xi - v_1\xi^2 \\ v_4 + 3v_3\xi + 3v_2\xi^2 - v_1\xi^3 \\ v_5 \end{pmatrix}$$

First assume $v_1 \neq 0$. Then we can solve the quadratic equation $v_3+2v_2\xi-v_1\xi^2 = 0$. Then $y = g_0(\xi)(v)$in U_3 has the coordinates $(y_1,\ldots,y_5)$ with $y_1 = v_1 \neq 0$ and $y_3 = 0$. Note that $f_3(y) = 0$ because f_3 is invariant under B_u. If $y_4 = 0$, then $y \in U_2'$ as required. Therefore assume $y_4 \neq 0$. We then calculate $z = g_0(\eta)$ with $\eta = 2y_2/y_1$. Let $(z_1,\ldots,z_5)$ be the coordinates of z. Then $z_3 = 0$ by construction and z_4 satisfies

$$\begin{aligned}&z_4 = y_4 + 3y_3\eta + 3y_2\eta^2 - y_1\eta^3\\&= y_1^{-2}(y_1^2y_4 + 4y_2^3) = y_1^{-2}y_4^{-1}f_3(y) = 0\ .\end{aligned}$$

This proves that $z \in U_2'$.

Otherwise $v_1 = 0$ and $v_2 \neq 0$. First assume $\mathrm{char}(K) \neq 2$. For $\xi = -v_3/2v_2$, the vector $y = g_0(\xi)v$ in U_3 has the coordinates $(y_1,\ldots,y_5)$ with $y_3 = 0$ and

$$\begin{aligned}&y_4 = v_4 + 3v_3\xi + 3v_2\xi^2\\&= (4v_2^2)^{-1}(4v_2^2v_4 - 3v_2v_3^2) = (4v_2^3)^{-1}f_3(v) = 0\ .\end{aligned}$$

This proves that $y \in U_2'$.

It remains the case that $\mathrm{char}(K) = 2$ and $v_2 \neq 0 = v_1$. We then observe that $0 = f_3(v) = v_2^2v_3^2$. This implies that $v_3 = 0$. If ξ solves the quadratic equation $v_4 + 3v_2\xi^2 = 0$, then $g_0(\xi)v \in U_2'$. $\square$

Remark. Alternatively, one can prove that C_3 and U_3^o are disjoint by showing that C_3 is irreducible of dimension 4, and that $U_3 \setminus U_3^o$ is closed and has dimension ≥ 4. The above proof is more explicit and illustrates Proposition 8. $\square$

4.4 Almost all strata are orbits

The Lemmas 21 and 22 show that the each of the dominant blades of $Nc(\mathfrak{g})$ is an orbit under its associated parabolic group, except for some cases in characteristic 2 and 3. The remaining cases are treated here, as well as the conclusions.

Lemma 23 *The Borel group B has a transitive action on U_4^o.*

Proof. For $v \in U_4^o$, say $v = \sum_{i=0}^5 v_iX_i$, we claim that there is an element $g \in B$ with $\mathrm{Ad}(g)v = X_0 + X_1$. Lemma 22 implies $v_0 \neq 0 \neq v_1$. We now use that $B = T \cdot B_u$ where T is a maximal torus of B and B_u is the unipotent subgroup of B. It is easy to see that there is $t \in T$ such that $\mathrm{Ad}(t)v \in X_0 + X_1 + \sum_{i=2}^5 KX_i$. It therefore suffices to show that B_u has a transitive action on $X_0 + X_1 + \sum_{i=2}^5 KX_i$.

We may therefore assume $v_0 = v_1 = 1$. In terms of the parametrization of B_u of Section 3.3, the equation $\mathrm{Ad}(b)(X_0 + X_1) = v$ with $b \in B_u$ is equivalent to the system of equations

$$\begin{aligned}&-u_0 + u_1 = v_2\\&-u_0^2 - 2u_2 = v_3\\&-u_0^3 - 3u_3 = v_4\\&-u_0^3u_1 - 3u_0^2u_2 + 3u_0u_3 - 3u_1u_3 - 3u_2^2 - u_4 = v_5\ .\end{aligned}$$

If the field K has characteristic $\neq 2$, 3, one can take $u_0 = 0$ and solve u_1, …, u_4 in a unique way.

Otherwise, the characteristic of K is 2 or 3. As K is algebraically closed, it is perfect. If K has characteristic 2, one first solves the equation $u_0^2 = -v_3$, puts $u_2 = 0$, and subsequently solves u_1, u_3, and u_4. If K has characteristic 3, one first solves the equation $u_0^3 = -v_4$, puts $u_3 = 0$, and solves u_1, u_2, and u_4. $\square$

It follows that the elements of U_4^o are regular in the sense of [2, p. 227].

Lemma 24 *The blade U_3^o is a single P_3-orbit.*

Proof. By Lemma 21(b), it remains to treat the case of $\mathrm{char}(K) = 3$. For this purpose, we use the subgroup H of G, the image of the homomorphism $h_0 : \mathbb{SL}(2) \to G$ considered in Section 3.3. This group H is a subgroup of the parabolic group P_3. Therefore, U_3 is an H-module. Indeed, as an H-module, it is a direct sum of the H-modules Q, the span of X_1, X_2, X_3, X_4, and the trivial H-module KX_5.

We first determine the H-orbit of the point $X_1 + X_3$ of Q, using $\mathrm{char}(K) = 3$. An element $a = \sum_{i=1}^4 a_i X_i$ satsfies

$$a = h_0 \begin{pmatrix} x & z \\ y & t \end{pmatrix} (X_1 + X_3)$$

if and only if there exist numbers x, y, z, t with

$$\begin{aligned} & xt - yz = 1 \\ & a_1 = t^3 \\ & a_2 = -t^2 y - z \\ & a_3 = -ty^2 + x \\ & a_4 = -y^3 \ . \end{aligned}$$

As K is perfect, the Frobenius mapping $x \mapsto x^3$ is an automorphism of the field K. The system of equations is therefore equivalent to $x^3 t^3 - y^3 z^3 = 1$ where $t^3 = a_1$, $y^3 = -a_4$, $x^3 = a_3^3 + a_1 a_4^2$, $z^3 = -a_2^3 + a_1^2 a_4$, or equivalently

$$-a_1^2 a_4^2 + a_1 a_3^3 - a_2^3 a_4 = 1 \ .$$

This is the equation $f_3(a) = -1$ because $\mathrm{char}(K) = 3$. This proves that the H-orbit of $X_1 + X_3$ is the subset of Q where $f_3 = -1$.

Let T_0 be the kernel of α_0 in torus T. Then $L_3 = T_0 H$ is a Levi subgroup of the parabolic group P_3. The adjoint action of T_0 multiplies all elements of Q with the same nonzero constant. Therefore the L_3-orbit of $X_1 + X_3$ is the subset of Q where $f_3 \neq 0$.

Finally, let w be an arbitrary element of U_3^o, say $w = q + \xi X_0$ with $q \in Q$ and $\xi \in K$. Then $f_3(q) = f_3(w) \neq 0$. Therefore w has a conjugate under L_3 of the form $X_1 + X_3 + \eta X_0$ with $\eta \in K$. The one-parameter subgroup g_0 conjugates this to $X_1 + X_3$. □

Using Lemma 11, Theorem 20, and the Lemmas 21, 23, 24, we obtain

Theorem 25 *(a) If $\mathrm{char}(K) \neq 3$, each of the strata of $\mathrm{Nc}(\mathfrak{g}_2)$ is a single G_2-orbit. (b) Assume $\mathrm{char}(K) = 3$. Each of the strata $G_2 U_i^o$ with $i \neq 2$ is a single G_2-orbit; the stratum $G_2 U_2^o$ is the union of two orbits: $G_2 U_{2a}$, $G_2 U_{2b}$ with $\dim(G_2 U_{2a}) = 8$ and $\dim(G_2 U_{2b}) = 6$.*

These five nilpotent orbits (or six if $\mathrm{char}(K) = 3$) correspond to the classes given in Table 22.1.5 of [13].

Moreover, as is easily verified, in each case, the adjacency structure of the orbits is the trivial one: orbit $\mathcal{O}'$ is contained in the closure of a different orbit $\mathcal{O}$ if and only if $\dim(\mathcal{O}') < \dim(\mathcal{O})$. This orbit structure corresponds with the results of [19].

The sizes of the Jordan blocks of representatives of the orbits (as matrices in $\mathfrak{sl}(C)$) are most easily obtained by calculating the ranks of the powers of the representative. If $\mathrm{char}(K) \neq 2$, the regular orbit has the sequence of sizes $(7, 1)$, the subregular orbit has $(3, 3, 1, 1)$. The next orbit has $(3, 2, 2, 1)$, followed by $(2, 2, 1^4)$, and finally (1^8). In characteristic 3, both orbits GU_{2a} and GU_{2b} have the same sequence $(3, 2, 2, 1)$. For characteristic 2, the sequences are $(4, 4)$, $(3, 3, 1, 1)$, (2^4), $(2, 2, 1^4)$, and (1^8).

5 The Nilpotent Variety and its Singularities

Let G be a reductive algebraic group with Lie algebra $\mathfrak{g}$. The starting point of the paper [7] was the question whether the G-orbits in $Nilp(G)$ can be classified using only the local structure of $Nilp(G)$. The paper gave a positive answer for the cases that G is $\mathbb{GL}(n)$ or $\mathbb{S}p(n)$ and $\mathrm{char}(K) \neq 2$. This result is extended here to the group G_2 in characteristic $\neq 2$, 3.

In order to deal with its local structure, we need to know $Nilp(G)$ as a subvariety of $\mathfrak{g}$. The next step is to introduce cross sections to investigate the local structure of $Nilp(G)$ at specific points. Smooth equivalence is introduced to formalize the idea of local structure, the criterion ord* serves to quantify it. In Section 5.4, cross sections are used to determine ord* of $Nilp(G)$ at the points of the five orbits in charateristics $\neq 2$ and 3. Characteristics 3 and 2 are treated in Sections 5.5 and 5.6, respectively.

5.1 The definition of the nilpotent variety

Let G be a reductive algebraic group over a field K with Lie algebra $\mathfrak{g}$. The affine quotient $[\mathfrak{g}/G]$ is the spectrum of the ring $A(\mathfrak{g})^G$ of the polynomial functions on $\mathfrak{g}$ that are invariant under the adjoint action of G. Let $p : \mathfrak{g} \to [\mathfrak{g}/G]$ be the canonical projection. The *nilpotent variety* $Nilp(G)$ is defined as the fiber $p^{-1}(p(0))$, see [7, (2.4)] (note that the affine quotient is universal because K is a field). This means that the defining equations of $Nilp(G)$ in $\mathfrak{g}$ are the homogeneous invariant polynomials of positive degree. In fact, by the argument at the end of Section 2.2, we do the same for the nullcone of any affine G-variety.

Let T be a maximal torus of G, with Lie algebra $\mathfrak{t}$ and Weyl group W. The restriction function $r : A(\mathfrak{g})^G \to A(\mathfrak{t})^W$ is injective and, under weak assumptions, it is an isomorphism [2, p. 200]. This holds in particular for the group G_2 in all characteristics. In Section 5.6 below it is shown for $\mathrm{char}(K) = 2$.

If $\mathrm{char}(K) \neq 2$ then r is an isomorphism for G_2 because of [7, (2.6)]. In this case, the ring $A(\mathfrak{g})^G$ is a graded polynomial algebra generated by algebraically independent homogeneous polynomials f_2, f_6 of degrees 2 and 6 [5, p. 296].

Now recall that the *characteristic polynomial* of an endomorphism x of a vector space V is defined as the polynomial in the indeterminate T given by $\chi(x) = \det(T{\cdot}\mathrm{id}-x)$. If $\dim V = n$, the *symmetrical polynomials* are the coefficients σ_i given by $\chi(x) = T^n + \sum_{i=1}^{n} \sigma_i(x) \cdot T^{n-i}$. Each coefficient σ_i is a homogeneous polynomial of degree i in the matrix coefficients of x, and is invariant under conjugation by elements of $\mathbb{GL}(V)$. The endomorphism x is nilpotent if and only if $\sigma_i(x) = 0$ for all $1 \leq i \leq n$.

We now specialize to the elements of the Lie algebra $\mathfrak{g}_2$ as endomorphisms of the algebra V introduced in the Section 3.2. In this case, $\sigma_i = 0$ for all odd indices i, and $\sigma_8 = 0$. The only nonzero symmetrical polynomials are σ_2, σ_4, and σ_6. It is convenient to introduce the polynomial

$$\tau_2 = v_0v_{13} + v_1v_{12} + v_2v_{11} + v_3v_{10} + v_4v_9 + v_5v_8 + v_6^2 + v_6v_7 + v_7^2 \ ,$$

which satisfies $\sigma_2 = 2\tau_2$ and $\sigma_4 = \tau_2^2$.

Theorem 26 *Assume that* $\mathrm{char}(K) \neq 2$. *The ring of G-invariant polynomials* $A(\mathfrak{g})^G$ *on* $\mathfrak{g}$ *is generated by* τ_2 *and* σ_6*, and these two generators are algebraically independent.*

Proof. Above we saw that $A(\mathfrak{g})^G$ is generated by homogeneous algebraically independent elements f_2 and f_6 of degrees 2 and 6. As τ_2 and σ_6 are homogeneous of degrees 2 and 6, there are scalars s_1, s_2, s_3 with $\tau_2 = s_1 f_2$ and $\sigma_6 = s_2 f_2^3 + s_3 f_6$.

It remains to show that s_1 and s_3 are nonzero. Well, $s_1 \neq 0$ because $x = X_1 + X_{12}$ satisfies $\tau_2(x) = 1$. Similarly, $s_3 \neq 0$ because $x = X_0 + X_1 + X_8$ satisfies $\tau_2(x) = 0$ and $\sigma_6(x) = 4 \neq 0$. □

5.2 Smooth equivalence and cross sections

If V is an algebraic variety over a field K, and $v \in V$, then the pair (V, v) is called a pointed variety. Pointed varieties (X, x) and (Y, y) are said to be smoothly equivalent, notation $(X, x) \sim (Y, y)$, iff there is a pointed variety (Z, z) with smooth morphisms $f : Z \to X$ and $g : Z \to Y$ such that $f(z) = x$ and $g(z) = y$. This is an equivalence relation between pointed varieties.

Let the group G act on a variety V by means of a morphism $h : G \times V \to V$. Let X be a subvariety of V, and $x \in X$. Then X is called a *cross section* at x if the restriction $h : G \times X \to V$ is smooth in the point $(1, x)$, see e.g. [7, Section 2]. As the group G is a smooth variety, it follows that $(X, x) \sim (V, x)$.

Now assume that V is a G-module. The action of G on V induces an action of the Lie algebra $\mathfrak{g}$ of G on V. Let L be a linear subspace of V. The restriction $h : G \times (x + L) \to V$ induces a tangent morphism $dh : \mathfrak{g} \times L \to V$ given by $dh(u, v) = u \cdot x + v$. Therefore $x + L$ is a cross section at x if and only if $\mathfrak{g} \cdot x + L = V$.

If $\mathrm{char}(K) \neq 2$ and 3, a cross section can be used in the following alternative proof of Theorem 26. Let $x = X_0 + X_1$ in $\mathfrak{g}_2$. The subspace $[\mathfrak{g}_2, x]$ is spanned by the vectors X_0, X_1, X_2, $2X_3$, $3X_4$, X_5, X_6, X_7, X_9, X_{10}, $2X_{11}$, $3X_{12} - X_{13}$. As $\mathrm{char}(K) \neq 2, 3$, the subspace L spanned by X_8 and X_{12} satisfies $[\mathfrak{g}_2, x] \oplus L = \mathfrak{g}_2$. Therefore $x + L$ is a cross section. By [7, Prop 2.2], the natural mapping $A(\mathfrak{g})^G \to A(x + L)$ is injective. Use the obvious coordinates v_8 and v_{12} on $x + L$. Then the invariant polynomials satisfy $(\tau_2 \mid x + L) = v_{12}$ and $(\sigma_6 \mid x + L) = 4v_8$. It follows that the subring $K[\tau_2, \sigma_6]$ generated by the two invariant polynomials is mapped bijectively to $A(x + L)$, that $K[\tau_2, \sigma_6]$ equals $A(\mathfrak{g})^G$, and that τ_2 and σ_6 are algebraically independent.

5.3 The sequence ord

The singularity of (V, v) can be characterised by a partition $\mathrm{ord}(V, v)$ defined as follows. Assume that A is the local ring of V at the point v. Assume that A is isomorphic to a quotient R/J where R is a regular local ring and J is an ideal of R. Let M be the maximal ideal of R. Then $\mathrm{ord}(V, v) = \mathrm{ord}(A)$ is the sequence of numbers $\mathrm{ord}^i(A) = \mathrm{rg}_{R/M}((J \cap (M^{i+1} + MJ))/MJ)$ for $i \geq 1$. It is proved in [7] that this definition does not depend on the choices made, and that $\mathrm{ord}(X, x) = \mathrm{ord}(Y, y)$ if $(X, x) \sim (Y, y)$. The sequence ord* is a descending sequence of natural numbers, almost all of them 0. It is often represented by the finite sequence of its positive elements.

Lemma 27 *Let R be a regular local ring with maximal ideal M. Let $A = R/J$ for some ideal $J \subset M$. Let r, s be natural numbers with $1 \leq r < s$.*
(a) Let J be generated by some element $f \in M^r \setminus M^{r+1}$. Then $\mathrm{ord}^i(A) = 1$ for $1 \leq i < r$, otherwise 0.
(b) Let J be generated by elements f, g with $f \in M^r \setminus M^{r+1}$ and $g \in M^s \setminus (fM \cup M^{s+1})$. Then $\mathrm{ord}^i(A) = 2$ for $1 \leq i < r$, and $\mathrm{ord}^i(A) = 1$ for $r \leq i < s$, and $\mathrm{ord}^i(A) = 0$ otherwise.

Proof. Part (a) can be left to the reader. (b) The vector space J/MJ over the field R/M is generated by the residue classes of f and g. The subspace $(J \cap (M^{i+1} + MJ))/MJ$ contains the class of f iff $i < r$; it contains the class of g iff $i < s$. Therefore, the classes are linearly independent, and the ranks are as described. □

5.4 Cross sections at the nilpotent elements for G_2

Cross sections are used to investigate the local structure of the nilpotent variety. Assume char$(K) \neq 2$, 3.

For every conjugacy class of nilpotent elements, we need to consider only one representative. We use the representatives given in the table in Section 4.1.

The cross section $x + L$ for the regular element $x = X_0 + X_1$ is constructed in Section 5.2 in the alternative proof of Theorem 26. It shows that x is a smooth point of $Nilp(G)$ with $\mathrm{ord}^* = ()$, i.e., $\mathrm{ord}^i = 0$ for all i.

Next the subregular element $x = X_1 + X_4$. The subspace $[\mathfrak{g}, x]$ is spanned by the vectors $X_i (0 \leq i < 8)$, and X_{13}, $X_9 - X_{12}$. A minimal cross section L is spanned by X_8, X_9, X_{10}, X_{11}. The restrictions of the symmetrical polynomials to $x + L$ are $\tau_2 = v_9$ and $\sigma_6 = v_8^2 - 4v_9v_{10}v_{11} + 4v_{10}^3 - 4v_{11}^3$. After elimination of v_9, it results the Kleinian singularity with the equation $v_8^2 + 4v_{10}^3 - 4v_{11}^3 = 0$. The singularity is at the origin, so the maximal ideal M is generated by v_8, v_{10}, v_{11}. As the lowest order term is quadratic, Lemma 27(a) gives $\mathrm{ord}^* = (1)$.

For the short root vector X_3, the subspace $[\mathfrak{g}, X_3]$ is spanned by X_0, X_2, X_3, X_4, X_5, $X_6 + X_7$, X_{11}, X_{13}. We can take L spanned by X_1, X_7, X_8, X_9, X_{10}, X_{12}. The restriction of the symmetrical polynomial τ_2 is $(\tau_2 \mid x + L) = v_1v_{12} - v_7^2 + 3v_{10}$. The restriction $(\sigma_6 \mid x + L)$ is rather messy. The variable v_{10} is eliminated by putting $(\tau_2 \mid x + L) = 0$. Then $(\sigma_6 \mid x + L)$ modulo M^4 equals

$$4(v_1v_8^2 + v_7v_8v_9 + v_9^2v_{12}) \ .$$

It follows that $\mathrm{ord}^* = (1, 1)$. In fact, one can verify that this is the singularity CC_3 described in [7, Section (4.5)] which also occurs the the nilpotent variety of the group of type C_2.

The long root vector X_5 gives the subspace $[\mathfrak{g}, X_5]$ spanned by X_1, X_2, X_3, X_4, X_5, X_6. Therefore the linear subspace spanned by the vectors X_0 and X_i with $7 \leq i \leq 13$ is a cross section. In this case $(\tau_2 \mid x + L) = v_0v_{13} - v_7^2 + 3v_8$. After elimination of v_8 by means of τ_2, the restriction $(\sigma_6 \mid x + L)$ modulo M^5 equals

$$-v_9^2v_{12}^2 - 6v_9v_{10}v_{11}v_{12} - 4v_9v_{11}^3 + 4v_{10}^3v_{12} + 3v_{10}^2v_{11}^2.$$

It follows that $\mathrm{ord}^* = (1, 1, 1)$. It can hardly be a coincidence that the function f_3 of Section 4.3 appears in this singularity.

In the case $x = 0$, the space $\mathfrak{g}$ itself is a minimal cross section. The singularity at the origin has $\mathrm{ord}^* = (2, 1, 1, 1, 1)$ by Lemma 27(b).

To summarize, this justifies the values of ord* of the singularities of $Nilp(G)$ in the five orbits given in the table of Section 4.1 when the characteristic of the field differs from 2 and 3. In particular, the singularities in the five orbits are different.

5.5 The local structure in characteristic 3

Now assume char$(K) = 3$. For the regular element $x = X_0 + X_1$, take the cross section $x + L$ where L is spanned by X_4, X_8, X_{12}. The intersection $(x + L) \cap Nilp(G)$ is given by the equations $v_{12} = 0$ and $v_8 - v_4^2v_8^2 = 0$. This is smooth at the origin, which corresponds to the point x. This shows that, also in this case, x is a smooth point of $Nilp(G)$ with $\mathrm{ord}^* = ()$.

For the subregular element $X_1 + X_4$, take the cross section $x + L$ where L is spanned by X_4, X_8, X_9, X_{10}, X_{11}. The intersection $(x + L) \cap Nilp(G)$ is given by the equations $v_4v_9 = 0$ and $-v_4^2v_8^2 + v_4^2v_{10}^3 - v_4v_{11}^3 = 0$, where the point x corresponds to $v_4 = 1$, $v_8 = v_9 = v_{10} = v_{11} = 0$. In particular, v_4 is invertible and the singularity has $\mathrm{ord}^* = (1)$, just as the subregular elements in Section 5.4.

Take $X_3 + X_5$ as a representative of the orbits G_2U_{2a} of Theorem 25(b). The image $[\mathfrak{g}_2, x]$ is spanned by X_0, X_2, X_3, X_4, X_5, $X_1 - X_{13}$, $X_6 + X_7$, $X_6 - X_{11}$. It follows that $x+L$ is a cross section at x if L is spanned by X_i with $i = 1, 6, 8, 9, 10, 12$. The intersection $(x+L)\cap \mathit{Nilp}(G)$ is given by the equations $v_6^2 = v_1 v_{12}$ and $-v_1^3 v_{12}^3 - v_1 v_6 v_9 v_{12}^2 + v_1 v_8^2 + v_1 v_8 v_{10} v_{12} - v_6 v_8 v_9 + v_8 v_{10}^2 - v_9^2 v_{12}^2 + v_9^2 v_{12} + v_{10}^3 v_{12} + v_{10}^3 = 0$. It follows that `ord*` $= (2,1)$ by Lemma 27(b).

One can take X_3 as a representative of the orbits $G_2 U_{2b}$ of Theorem 25(b). The formulas get more complicated. It is therefore unlikely that the point X_3 is smoothly equivalent to $X_3 + X_5$. Yet ord* $= (2,1)$, so ord* does not separate the orbits.

One can determine the singularity of $\mathit{Nilp}(G)$ at the point X_5 by the same method. It turns out that ord* $= (2,1,1)$. The singularity at the origin has ord* $= (2,1,1,1,1)$, just as in Section 5.4. Summarizing, the singularities in the five strata are separated by ord*, but the singularities in the orbits are not.

5.6 Invariant polynomials in characteristic 2

Assume that $\mathrm{char}(K) = 2$. In this case, the polynomial σ_2 vanishes but τ_2 is G_2-invariant because $\tau_2^2 = \sigma_4$ and, in characteristic 2, squaring is injective. Similarly, the polynomial

$$\begin{aligned}\tau_3 = {} & (v_6+v_7)(v_6v_7 + v_1v_{12} + v_3v_{10}) \\ & + v_6(v_2v_{11} + v_4v_9) + v_7(v_0v_{13} + v_5v_8) \\ & + v_0v_1v_{11} + v_5v_{10}v_{11} + v_4v_{10}v_{13} + v_2v_{12}v_{13} \\ & + v_2v_3v_8 + v_0v_3v_9 + v_1v_4v_8 + v_5v_9v_{12}\end{aligned}$$

satisfies $\tau_3^2 = \sigma_6$ and is therefore G_2-invariant. It is easy to see that $x \in \mathfrak{g}_2$ is nilpotent if and only if $\tau_2(x) = \tau_3(x) = 0$. Therefore, if $f(x) = 0$ for all nilpotent x, then some power of f is in the ideal generated by τ_2 and τ_3, by Hilbert's Nullstellensatz.

For the regular element $x = X_0 + X_1$, take the cross section $x + L$ where L is spanned by X_3, X_8, X_{11}, X_{12}. Now $(\tau_2 \mid x+L) = v_{12}$ and $(\tau_3 \mid x+L) = v_{11}$. The intersection $(x+L)\cap \mathit{Nilp}(G)$ is therefore given by the equation $v_{11} = v_{12} = 0$. This implies that x is a smooth point of $\mathit{Nilp}(G)$, as expected. It also follows that τ_2 and τ_3 are algebraically independent in $A(\mathfrak{g})^G$.

We now come back to the injective restriction $r : A(\mathfrak{g})^G \to A(\mathfrak{t})^W$ of Section 5.1. In characteristic 2, the Lie algebra $\mathfrak{t}$ consists of the matrices

$$\mathrm{diag}(v_{67}, v_6, v_7, 0, 0, v_7, v_6, v_{67}) \text{ with } v_{67} + v_6 + v_7 = 0.$$

The Weyl group is generated by the reflexions s_0 and s_1 in the simple roots α_0 and α_1. These reflexions act on $\mathfrak{t}$ as the adjoint actions of the group elements

$$w_0 = h_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } w_1 = h_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Here h_0 and h_1 are the homomorphisms $\mathbb{SL}(2) \to G_2$ introduced in Section 3.3. The element w_0 interchanges v_6 and v_{67}, while w_1 interchanges v_6 and v_7. This implies that the Weyl group acts as the permutation group of the symbols v_6, v_7, v_{67}, i.e., as the Weyl group of A_2 on the weight lattice. Therefore, the ring $A(\mathfrak{t})^W$ is generated by homogeneous polynomials of degrees 2 and 3 by [5, p. 296]. As the functions τ_2 and τ_3 in $A(\mathfrak{g})^G$ restrict to W-invariant homogeous polynomials of degrees 2 and 3, it follows that

Theorem 28 *Assume that $\mathrm{char}(K) = 2$. The restriction function $r : A(\mathfrak{g})^G \to A(\mathfrak{t})^W$ is an isomorphism. The ring of G-invariant polynomials $A(\mathfrak{g})^G$ on $\mathfrak{g}$ is generated by τ_2 and τ_3, and these two generators are algebraically independent.*

Above we saw that the regular elements are smooth points of *Nilp*(G). It turns out that the subregular elements are smooth points of *Nilp*(G) as well. Indeed, for the subregular element $X_1 + X_4$, take the cross section $x + L$ where L is spanned by X_8, X_9, X_{10}, X_{11}. The restrictions of τ_2 and τ_3 to $x + L$ are $\tau_2 = v_9$ and $\tau_3 = v_8$.

For the short root vector $x = X_3$, take the cross section $x+L$ where L is spanned by X_i with $i = 0, 1, 2, 7, 8, 9, 10, 12$. Then $(\tau_2 \mid x + L) = v_{10} + v_1 v_{12} + v_7^2 + v_{10}$ and $(\tau_3 \mid x + L) = v_0 v_9 + v_1 v_7 v_{12} + v_2 v_8 + v_7 v_{10}$. After elimination of v_{10}, this leads to the Kleinian singularity with the equation $v_0 v_9 + v_2 v_8 + v_7^3 = 0$ and $\mathrm{ord}^* = (1)$.

For the long root vector $x = X_5$, take the cross section $x+L$ where L is spanned by X_i with $i = 0$ or $7 \le i < 14$. Then $(\tau_2 \mid x + L) = v_8 + v_0 v_{13} + v_7^2$ and $(\tau_3 \mid x + L) = v_0 v_7 v_{13} + v_7 v_8 + v_9 v_{12} + v_{10} v_{11}$. After elimination of v_8, this leads to the equation $v_9 v_{12} + v_{10} v_{11} + v_7^3 = 0$. This gives the same singularity as at the short root vector.

The singularity at the origin has $\mathrm{ord}^* = (2, 1)$.

Acknowledgement. I am grateful to Gerhard Röhrle for helping me to come back to this field.

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